IM 9–12 MATH v.1

Unit 1
Constructions and Rigid Transformations

GEOMETRY

Lesson 19

Evidence, Angles, and Proof

Illustrative Mathematics®

Kendall Hunt

Learning Goal

Let’s make convincing explanations.

Geometry

Warm-up: Math Talk

Mentally evaluate all of the missing angle measures in each figure.

Figure A

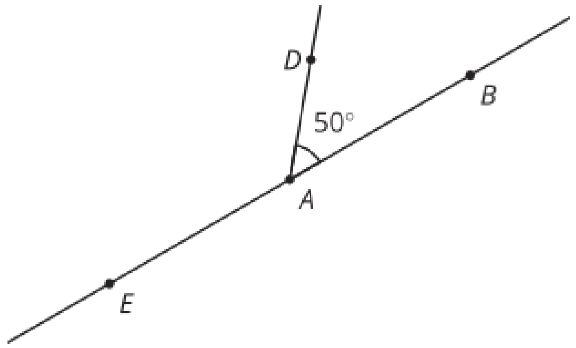


Figure B

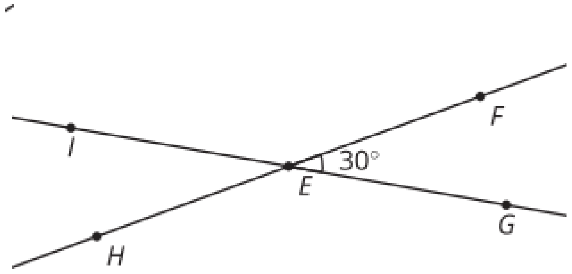


Figure C

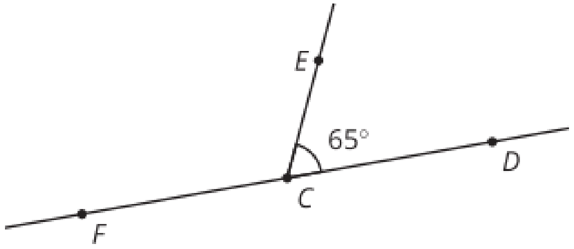
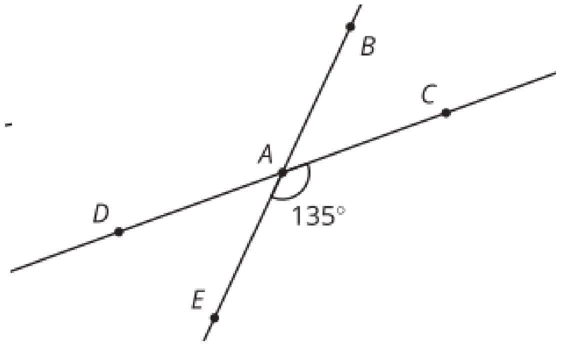


Figure D



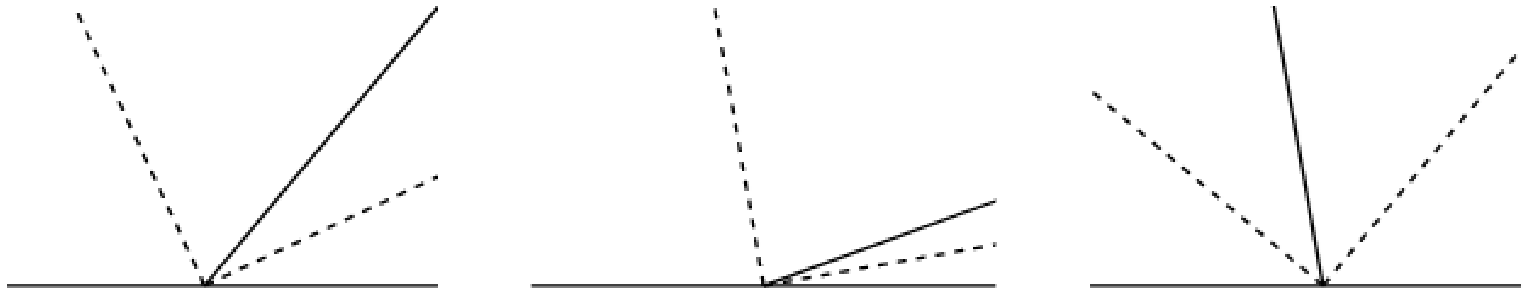
Supplementary Angles

Warm-up: Math Talk

That Can't Be Right, Can It?

Notice and Wonder

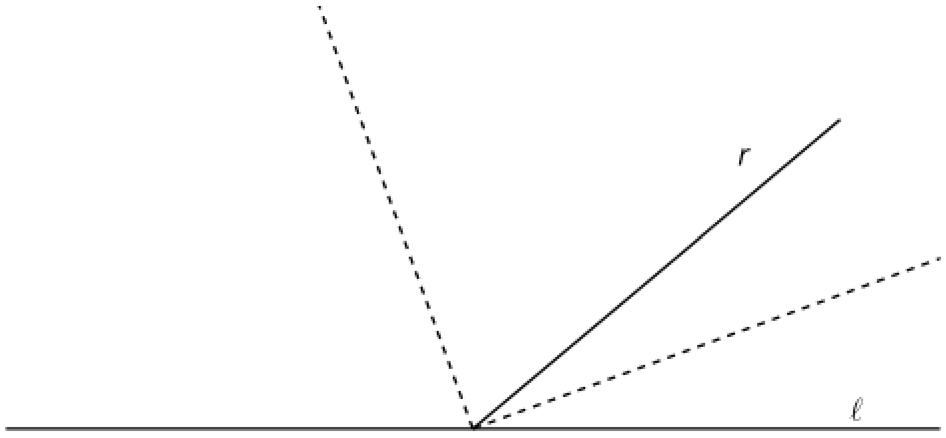
What do you notice? What do you wonder?



That Can't Be Right, Can It?

Here is a figure where ray *r* meets line *l*. The dashed rays are angle bisectors.

1. Diego made the conjecture: "The angle formed between the angle bisectors is always a right angle, no matter what the angle between *r* and *l* is." It is difficult to tell specifically which angles Diego is talking about in his conjecture. Label the diagram and rephrase Diego's conjecture more precisely using your labels.
2. Is the conjecture true? Explain your reasoning.

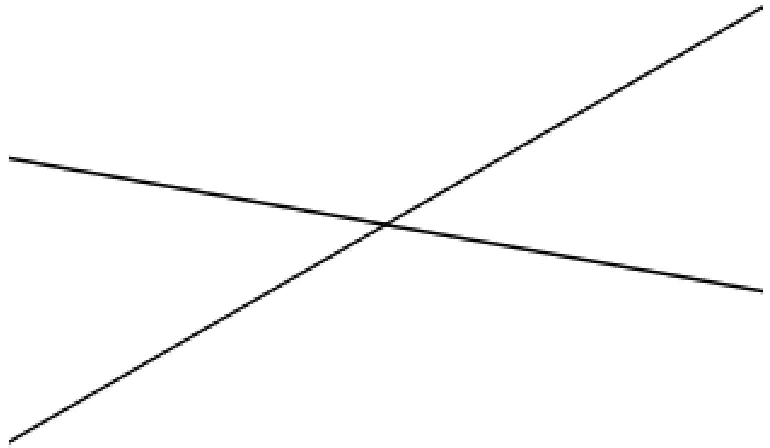


That Can't Be Right, Can It?

Convince Me

Here are 2 intersecting lines that create 2 pairs of vertical angles:

1. What is the relationship between vertical angles? Write down a conjecture. Label the diagram to make it easier to write your conjecture precisely.
2. How do you know your conjecture is true for all possible pairs of vertical angles? Explain your reasoning.

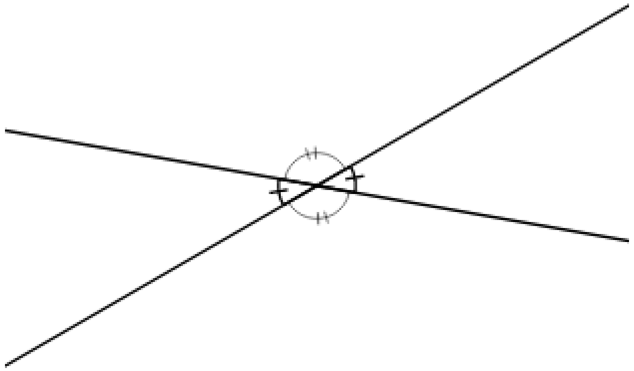


Convince Me

- Which argument makes more sense to you, rigid transformations that take one vertical angle onto the other, or using straight angles to look at 180 degree sums?
- What is the difference between angle and angle measure?

Ask students to add this theorem to their reference charts as you add it to the class reference chart:

Vertical angles are congruent.



- I can label and make conjectures from diagrams.
- I can prove vertical angles are congruent.

Learning Targets

Geometry

NAME DATE PERIOD

Lesson 19: Evidence, Angles, and Proof

Goals

- Label diagrams and explain conjectures (orally and in writing).
- Prove (in writing) that vertical angles are congruent.

Learning Targets

- I can label and make conjectures from diagrams.
- I can prove vertical angles are congruent.

Lesson Narrative

In previous grades, you learned a bit about angles—like supplementary, complementary, vertical, and adjacent angles—to solve problems. More recently, you've been making conjectures, creating definitions of rigid motions, and explaining why you believe certain statements are true or false. All of this work has been building toward a deeper understanding of geometry, and over the next few lessons, you'll start learning how to express your reasoning more formally.

Today, we're going to focus on making conjectures about angle relationships, like vertical angles, and proving them using what you know about rigid transformations. You'll also start labeling and marking figures to show congruence, which helps you explain your thinking more clearly and precisely. One of the big goals today is for you to write strong arguments and critique each other's reasoning, especially as you work to show why vertical angles are congruent. This is part of what mathematicians do all the time—making sure their explanations are convincing (that's Mathematical Practice 3).

The proofs we'll be working with are written in narrative form. Why narrative? Well, think about it this way: when you explain something to a partner, you're trying to tell a story that makes sense to them. The same goes for the way mathematicians write proofs—it's like telling a story where each step follows logically from the one before. You might be used to seeing proofs written in two-column format, and that's still useful for organizing your thoughts. But just like an outline is helpful for writing an essay, the real goal is to write the essay itself, the full argument. That's why you'll focus on narrative proofs—they show the flow of reasoning more clearly and persuasively. Plus, it's a great opportunity for you to make sense of the problem and keep pushing forward when things get tricky (that's Mathematical Practice 1).

NAME DATE PERIOD

19.1 Math Talk: Supplementary Angles

Warm Up: 5 minutes

"The purpose of today's Math Talk is to get you thinking about strategies for finding angle measures when we have pairs of intersecting lines or angles that form a straight angle. These kinds of problems build on what you already know about angle relationships—like how certain angles add up to 180 degrees or how vertical angles are related. The ideas you share today will help you develop fluency with these concepts, which is going to be really important for the rest of today's lesson.

Later on, you'll need to explain why vertical angles are congruent, so this is a great opportunity to warm up your thinking. As you work through the Math Talk, I want you to pay attention to the structure of the angle relationships you see—that's where you can really make use of Mathematical Practice 7. When you identify angles that are supplementary, or when you notice relationships between angles formed by intersecting lines, you're using structure to help solve the problem.

Remember, Math Talks are about sharing strategies and learning from each other. There's no one right way to approach these problems, so if you have a different method than someone else, share it! Your strategies are valuable, and they'll help all of us think more deeply about angle relationships."

Launch

Display one problem at a time. Give students quiet think time for each problem and ask them to give a signal when they have an answer and a strategy. Keep all problems displayed throughout the talk. Follow with a whole-class discussion.

Representation: Internalize Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organization

Student Task Statement

Mentally evaluate all of the missing angle measures in each figure.

NAME DATE PERIOD

Figure A

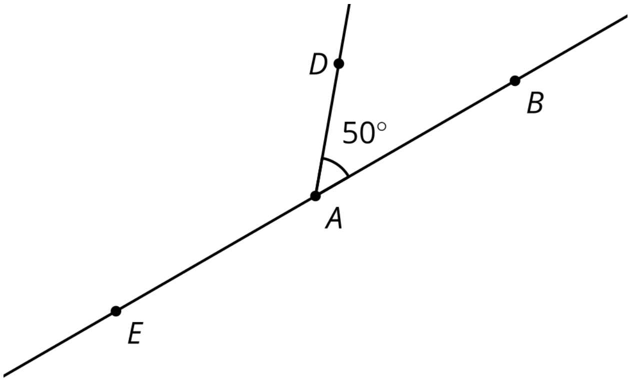


Figure B

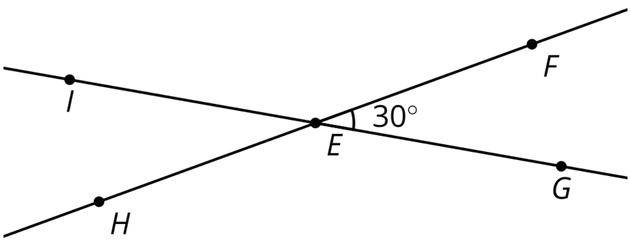


Figure C

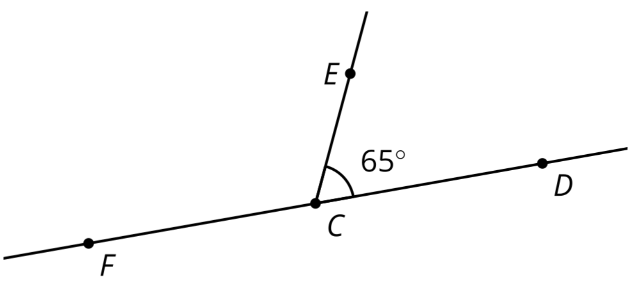
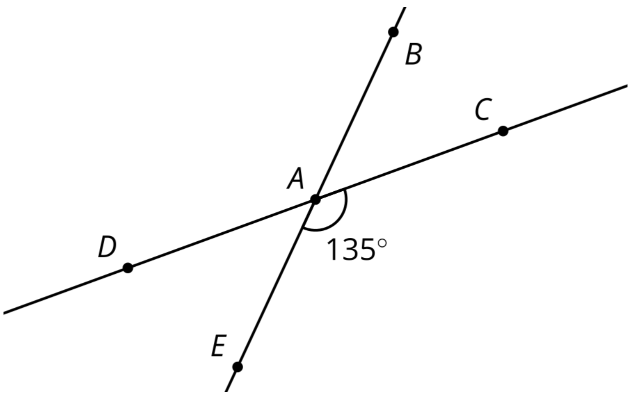


Figure D



Student Response

- Figure A: $m\angle DAE = 130°, m\angle EAB = 180°$
- Figure B: $m\angle FEI = 150°, m\angle HEI = 30°, m\angle HEG = 150°$
- Figure C: $m\angle ECF = 115°, m\angle FCD = 180°$
- Figure D: $m\angle BAD = 135°, m\angle EAD = 45°, m\angle BAC = 45°$

Activity Synthesis

Who would like to start by sharing how they solved the first problem? Remember, there's more than one way to think about these, so don't hesitate to speak up.

[After a student shares] Great! Can someone restate [Student]'s reasoning in a different way? How would you explain their strategy?

Did anyone have the same strategy but explain it differently, or maybe used different steps? I'd love to hear another way of thinking about it.

Who solved the problem in a completely different way? Let's add that perspective to the mix too.

[If a student shares something new] Does anyone want to build on or add to [Student]'s strategy? Maybe there's a connection or detail we can include.

And finally, do you agree or disagree with the reasoning that's been shared so far? Why do you think it works—or why not? Let's really dive into why these strategies are sound."

NAME DATE PERIOD

19.2 That Can't Be Right, Can It?

15 minutes

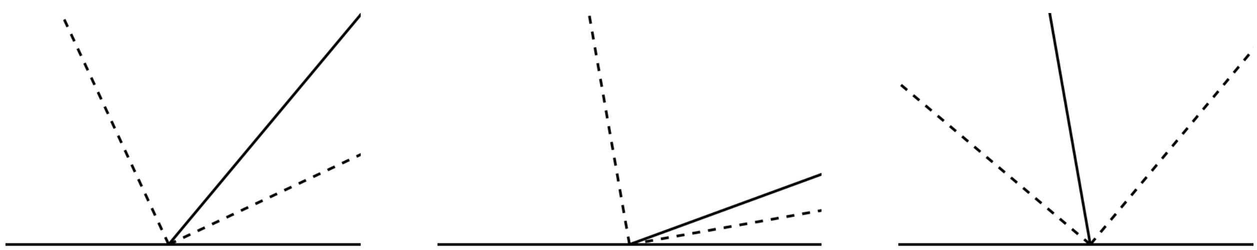
"The purpose of today's task is to take an idea—or a conjecture—that you've been working with informally and describe it more precisely by labeling a figure. You're going to start by looking at three examples of angle bisectors, and from those examples, you'll form a conjecture about what you notice.

This step is really important because it helps you slow down and make sense of the problem before jumping straight into solving it. This connects to Mathematical Practice 1—taking a step back, getting familiar with the context, and thinking about the mathematics involved. When you begin labeling the diagram, you might have different ways of marking the angles or the bisectors, and that's okay. The goal is to use the labels to help you make sense of what's happening and explain your thinking clearly.

As we move through the task, we'll refine how we describe our conjectures more precisely during the discussion. And remember, if you want to explore this further or try out different ideas, dynamic geometry software is available to help you make sense of the relationships and choose tools strategically (Mathematical Practice 5). So, let's dive in, and don't hesitate to use all the resources at your disposal to help you figure out the task!"

Launch

Display three examples of angle bisectors of linear pairs for all to see:



Ask students, "What do you notice? What do you wonder?"

Things students may notice:

- there are solid and dashed lines
- there is a horizontal line in all three
- the two solid angles make a linear pair

Things students may wonder:

- Are the dashed lines angle bisectors?
- Do the dashed lines make a right angle?

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. If the conjecture that the angle between the angle bisectors is always a right angle does not come up during the conversation, ask students to discuss this idea.

If students have access to GeoGebra Geometry from Math Tools, suggest that it might be a helpful tool in this activity.

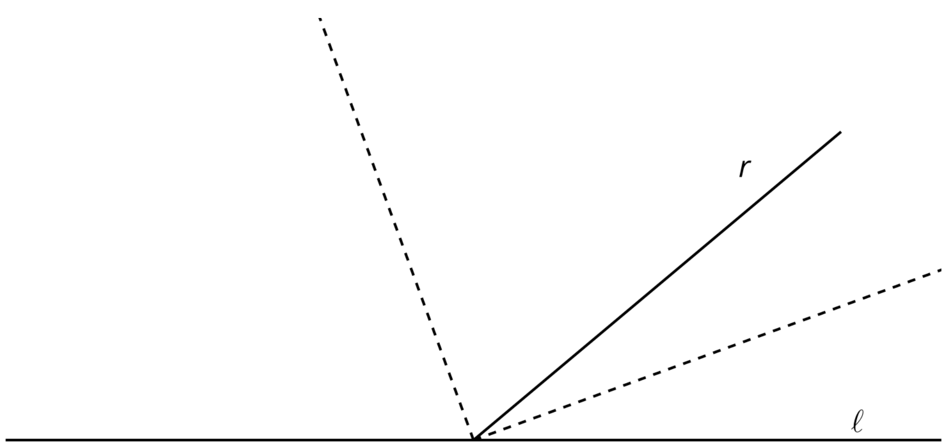
Action and Expression: Internalize Executive Functions. Provide students with a table to record what they notice and wonder prior to being expected to share these ideas with others.
Supports accessibility for: Language; Organization

Anticipated Misconceptions

If students get stuck, ask them to estimate the measure of one angle and then make arguments based on angle measure like in the warm-up. If time allows, invite students to generalize.

Student Task Statement

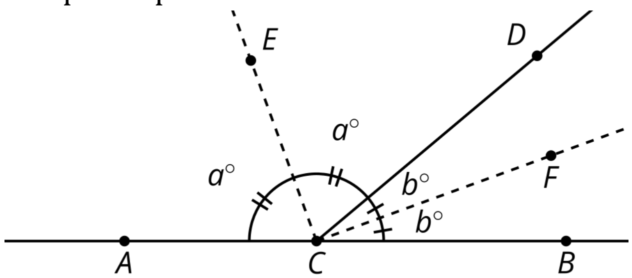
Here is a figure where ray r meets line ℓ. The dashed rays are angle bisectors.



1. Diego made the conjecture: "The angle formed between the angle bisectors is always a right angle, no matter what the angle between r and ℓ is." It is difficult to tell specifically which angles Diego is talking about in his conjecture. Label the diagram and rephrase Diego's conjecture more precisely using your labels.
2. Is the conjecture true? Explain your reasoning.

Student Response

1. Sample response:



   With this labeling, Diego's conjecture can be restated as: "Given that ray CE bisects angle ACD into two congruent angles that measure $a°$ and that ray CF bisects angle BCD into two congruent angles that measure $b°$, then angle ECF is a right angle. In other words, $a + b = 90$."

2. Sample response: Yes, Diego's conjecture is true. Angle ACB forms a line, and so measures 180 degrees. That means $2a + 2b = 180$. Dividing each side by 2, we have $a + b = 90$. The expression $a + b$ can be interpreted to represent the measure of angle ECF, and so angle ECF is a right angle. This argument works no matter the particular values of a and b, so long as those values make sense with the constraints of the figure.

Activity Synthesis

Now that we've worked through the task, it's time to discuss what we've discovered and introduce some important concepts about marking angles as congruent, labeling points, and labeling angles clearly.

Let's start by talking about Diego's conjecture. Who can share a convincing argument about whether Diego's conjecture is true or false? As you explain, I'll label an image so we can all see the information you're providing. Remember, you don't need to write a formal proof yet, but I encourage you to rephrase your reasoning using more precise mathematical language as we go.

As you share how you labeled the figure, we'll take this opportunity to build on those ideas and talk about some key conventions we use in geometry. We'll introduce how to mark angles as congruent, label points, and show variable angle measures. For example, here's one way we could mark the figure, similar to a sample student response.

The goal here is to make sure we all have a common way of labeling and marking figures so that our explanations are as clear and precise as possible. So, let's dive into the details and solidify these conventions together.

19.3 Convince Me

15 minutes

"The purpose of today's activity is for you to prove that vertical angles are congruent and work towards expressing your reasoning in a more formal and rigorous way. You've already explored angle relationships and transformations, so now it's time to take what you know and apply it to building a strong argument using these ideas.

As you work through this task, I want you to remember to label points and make markings on the diagram to help explain your thinking more clearly. This will make it easier for both you and others to follow your reasoning as you share your ideas.

I'll be walking around and listening for different types of arguments. Some of you might base your proof on rigid transformations, while others may focus on supplementary angles to explain why vertical angles are congruent. Both approaches are valuable, so feel free to think about it in your own way and use the tools that make the most sense to you.

Let's get started, and remember, the goal is to not only prove that vertical angles are congruent but also to become more precise and formal in the way you communicate your ideas."

Launch

Display two intersecting lines for all to see. Remind students that the pairs of angles opposite the intersection point are called vertical angles. Arrange students in groups of 2. Tell students there are many possible answers for the questions. After quiet work time, ask students to compare their responses to their partner's and decide if they are both correct, even if they are different. Follow with whole-class discussion.

Engagement: Internalize Self Regulation. Demonstrate giving and receiving constructive feedback. Use a structured process and display sentence frames to support productive feedback. For example, "How do you know…?," "That could/couldn't be true because…," and "We can agree that…."
Supports accessibility for: Social-emotional skills; Organization; Language

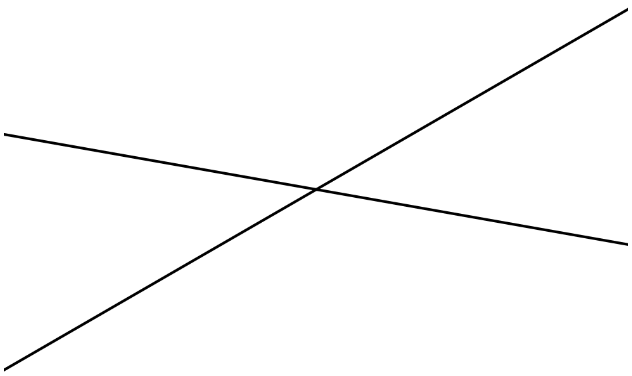
Anticipated Misconceptions

If students are stuck, suggest they label one of the acute angles as x°. Ask what else they can label or figure out based on that information.

NAME DATE PERIOD

Student Task Statement

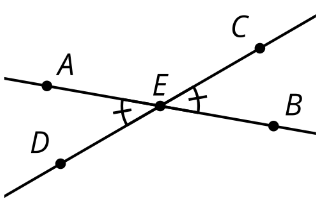
Here are 2 intersecting lines that create 2 pairs of vertical angles:



1. What is the relationship between vertical angles? Write down a conjecture. Label the diagram to make it easier to write your conjecture precisely.

2. How do you know your conjecture is true for all possible pairs of vertical angles? Explain your reasoning.

Student Response



1. Vertical angles are congruent. Angle AED is congruent to angle BEC.
2. Sample response:
   Rotate the figure 180 degrees around point E. Then ray EA goes to ray EB and ray ED goes to ray EC. That means the rotation takes angle AED onto angle BEC, and so $\angle AED \cong \angle BEC$. None of those statments are specific to these particular angles. If the sizes of the angles changed, nothing about the reasoning would change.

Are You Ready for More?

One reason mathematicians like to have rigorous proofs even when conjectures seem to be true is that sometimes conjectures that are made turn out to not be true. Here is one famous example. If we draw n points on a circle and connect each pair of points how many regions does that divide the circle into? If we draw only 1 point there are no line segments to connect and so just 1 region in the circle. If we draw 2 points they are connected by a line segment which divides the circle into 2 regions.

NAME DATE PERIOD

1. If we draw 3 points on a circle and connect each pair of points with a line segment how many regions do we get in our circle?
2. If we draw 4 points on a circle and connect each pair of points with a line segment how many regions do we get in our circle?
3. If we draw 5 points on a circle and connect each pair of points with a line segment how many regions do we get in our circle?
4. Make a conjecture about how many regions we get if we draw n points on a circle and connect each pair of points with a line segment.
5. Test your conjecture with 6 points on a circle. How many regions do we get?

Student Response

1. 4
2. 8
3. 16
4. Sample response: 2^{n-1}
5. 31

Activity Synthesis

"Let's come together now and refine our arguments into convincing proofs. We're going to take the ideas you've worked on and build them into something more formal that could be used to prove vertical angles are always congruent.

To make sure we're all on the same page, let's start by labeling the points on the diagram. I'll use your input to label these points so that when we discuss, we're all talking about the same objects in a consistent way.

Now, I'd like to hear from some of the students I've spoken with who focused on using transformations to prove that vertical angles are congruent. Could you share your reasoning with the class? Let's walk through that argument together.

Next, I'd like to invite those of you who used supplementary angles in your argument. Could you explain how you used the idea of supplementary angles to show vertical angles are congruent?

After hearing both arguments, let's ask an important question: Did either argument depend on the specific angles we were given in this example? Remember, for a proof to be valid, it must work for any vertical angles, not just this particular case. Otherwise, we're only working with an example, not a proof.

In the cool-down, you'll be writing an explanation about why vertical angles are congruent, so it's important that you understand at least one of these arguments. If you have any questions or if something wasn't clear, now is the time to ask so that everyone feels confident in the reasoning."

Lesson Synthesis

As we wrap up today's lesson, I want you to think about this: In everyday life, it's often complicated to fully understand why certain statements are true or false. Think about something like the economic system, international relations, or even the history of a country—there are so many factors at play, and it can be hard to grasp all the reasons behind things. But in geometry, the objects we study—angles, lines, points, triangles—are much simpler in comparison. This makes geometry a great training ground for understanding why ideas are true and how to communicate those reasons to others clearly.

Earlier in the lesson, you came up with different explanations for why vertical angles are congruent. Now, let's talk about the types of arguments we used. Some of you used angle measures to explain it, and some of you used rigid transformations. What's the difference between these two kinds of reasoning?

Which one makes more sense to you? Do you find rigid transformations, where you can see one vertical angle being mapped onto the other, more intuitive? Or do you prefer using straight angles and looking at how they add up to 180 degrees?

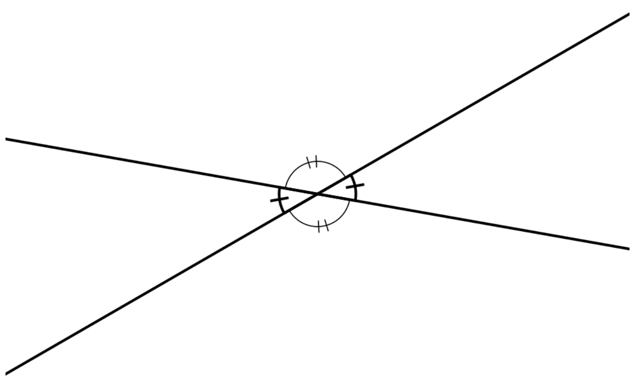
[Pause for student responses.]

And now, let's talk about the difference between an angle and its angle measure. How would you describe that difference? Think about it like the difference between a segment and its length. Segments and angles are geometric figures, but the length of a segment and the measure of an angle are numbers that describe how big or small they are.

[Pause for student responses.]

To wrap up, I want you to add a new theorem to your reference charts as I add it to ours: *Vertical angles are congruent.* This is something you've now proven, and it's a fundamental concept you'll be able to use again in future lessons."

(Theorem)



Provide the tip: Look for vertical angles whenever two lines intersect.